Analysis and Geometry of Fractals and Metric Spaces: Recent Developments and Future Prospects

5-9 March 2023

Okinawa, Bankoku Shinryokan, Ocean Hall

Organisers: Naotaka Kajino (Kyoto University, Chair), David Croydon (Kyoto University), Daisuke Shiraishi (Kyoto University)

Sunday 5th March	Monday 6th March	Tuesday 7th March	Wednesday 8th March	Thursday 9th March
	09:30–10:20 Kleiner	09:30–10:20 Ohta	09:10–10:00 Mackay	09:20–10:10 Chen
10:00–14:00 Registration	10:40–11:30 Bonk	10:40–11:30 Lapidus*	10:10–10:35 Sasaya	10:20–10:45 Shimizu
(12:00–14:00 Lunch)	11:40–12:30 Tsukamoto	11:40–12:30 Hambly	10:50–11:40 Tanaka	11:10–12:00 Saksman
14:00–14:50 Barlow	14:00–14:50 Murugan	14:00–14:50 Saloff-Coste		13:20–14:10 Miermont
15:10–16:00 Meyer	15:10–16:00 Grigor'yan*	15:10–16:00 Tyson		14:20–15:10 Kumagai
16:20–17:10 Shanmugalingam	16:20–17:10 Aikawa	16:20–17:10 Teplyaev		15:15–15:25 Closing

$*$ = online presentation.

Martin Barlow . . . . . *Forty years of analysis on fractals*
Daniel Meyer . . . . . *Quasisymmetric uniformization and quasivisual approximations*
Nageswari Shanmugalingam . . . . . *Constructing boundaries of domains in metric spaces using prime ends*
Bruce Kleiner . . . . . *Rigidity and flexibility of mappings between Carnot groups*
Mario Bonk . . . . . *Green functions in metric measure spaces*
Masaki Tsukamoto . . . . . *Introduction to mean dimension*
Mathav Murugan . . . . . *Quasisymmetry and the elliptic Harnack inequality*
Alexander Grigor'yan* . . . . . *Off-diagonal estimates of heat kernels for jump processes*
Hiroaki Aikawa . . . . . *Elliptic and parabolic boundary Harnack principles via Harnack inequality admitting exceptional set*
Shin-ichi Ohta . . . . . *On weighted Ricci curvature with negative dimension parameter*
Michel Lapidus* . . . . . *An introduction to the theory of complex dimensions*
Ben Hambly . . . . . *Spectral asymptotics for some domains with random fractal boundaries*
Laurent Saloff-Coste . . . . . *Limit theorems for stable-like random walks on nilpotent groups*
Jeremy Tyson . . . . . *On H-type and polarizable Carnot groups*
Alexander Teplyaev . . . . . *BV and harmonic functions on fractals*
John Mackay . . . . . *Poincaré inequalities and hyperbolic groups*
Kôhei Sasaya . . . . . *A system of dyadic cubes of a complete, doubling, uniformly perfect metric space without detours*
Ryokichi Tanaka . . . . . *Non-noise sensitivity for random walks on word hyperbolic groups*
Zhen-Qing Chen . . . . . *Convergence of effective resistances on generalized Sierpinski carpets*
Ryosuke Shimizu . . . . . *Non-linear potential theory on the Sierpiński carpet*
Eero Saksman . . . . . *GFF, chaos and Riemann zeta*
Grégory Miermont . . . . . *Compact Brownian surfaces*
Takashi Kumagai . . . . . *Spectral dimension of simple random walk on random media*

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Abstracts

Martin Barlow (University of British Columbia)

Forty years of analysis on fractals

It is now about 40 years since the first papers by mathematical physicists which studied statistical physics models on fractals, and introduced mathematicians to this topic. This talk will survey some parts of the history of the field of analysis and diffusions on fractals. It will also recall some unsolved problems and (largely) unexplored directions.

Daniel Meyer (University of Liverpool)

Quasisymmetric uniformization and quasivisual approximations

Quasisymmetric maps are homeomorphisms where ratios of distances are controlled. They are generalizations of conformal maps. Originating in complex analysis, they appear in a variety of contexts. In particular they appear in geometric group theory (at the boundary at infinity of Gromov hyperbolic groups). The purpose of this talk is to give an overview of quasisymmetric uniformization and to present a recent framework for this. Namely, "quasivisual approximations" provide a sequence of discrete approximations of spaces as well as necessary and sufficient conditions for quasisymmetric equivalence. This is closely related to recent work by Jun Kigami. The talk is based on joint work with Mario Bonk.

Nageswari Shanmugalingam (University of Cincinnati)

Constructing boundaries of domains in metric spaces using prime ends

To obtain continuous extensions of conformal mappings between planar domains to their boundaries, Carathéodory constructed a notion of prime ends for simply connected planar domains, and demonstrated that conformal maps between simply connected planar domains do extend as homeomorphisms to their respective prime end boundaries. For non-simply connected domains Carathéodory's construction need not work. In this talk we will discuss an alternative notion of prime ends and discuss its properties, with application to potential theory and branched quasisymmetric maps.

Bruce Kleiner (New York University)

Rigidity and flexibility of mappings between Carnot groups

The lecture will discuss some recent results concerning the regularity, rigidity, and flexibility of bilipschitz, quasiconformal, and more generally Sobolev mappings between Carnot groups. This is based on joint work with Stefan Muller, Laszlo Szekelyhidi, and Xiangdong Xie.

Mario Bonk (University of California, Los Angeles)

Green functions in metric measure spaces

Let X be an unbounded Ahlfors Q-regular Q-Loewner space with $Q \geq 2$. Then X admits a measurable differential structure in the sense of Cheeger and one can define a notion of "Cheeger harmonic" functions on X. In my talk I will discuss the statement that in this setting for every point $x_0 \in X$ there exists a Green function G_{x_0} with pole at x_0. It is harmonic in $X \setminus \{x_0\}$ and has logrithmic blow-up near x_0 and near ∞ (with different signs). With suitable normalization, this Green function G_{x_0} is unique. This is joint work with Luca Capogna and Xiaodan Zhou.

Masaki Tsukamoto (Kyoto University)

Introduction to mean dimension

Mean dimension is a topological invariant of dynamical systems introduced by Gromov in 1999. It evaluates the number of parameters per unit time for describing a given dynamical system. Gromov introduced this notion for the purpose of exploring a new direction of geometric analysis. Independently of this original motivation, Elon Lindenstrauss and Benjamin Weiss found deep applications of mean dimension in topological dynamics. I plan to survey some highlights of the mean dimension theory.

Mathav Murugan (University of British Columbia)

Quasisymmetry and the elliptic Harnack inequality

Jun Kigami made an elementary but useful observation that the elliptic Harnack inequality is preserved under a quasisymmetric change of metric. In this talk, I will survey three works that rely on this observation: a characterization of quasisymmetry of circle packing metric using heat kernel estimates and Harnack inequalities, the stability of the elliptic Harnack inequality, and the notion of conformal walk dimension. This talk is based on joint works with Martin Barlow, Zhen-Qing Chen, and Naotaka Kajino.

Alexander Grigor'yan* (Universität Bielefeld)

Off-diagonal estimates of heat kernels for jump processes

We prove upper bounds of the heat kernel for a jump type Dirichlet form on a doubling metric measure space, where the off-diagonal term depends on the L^q tail estimate of the jump kernel. An important ingredient of the proof is a parabolic mean value inequality. This is a joint work with Jiaxin Hu and Eryan Hu.

Hiroaki Aikawa (Chubu University)

Elliptic and parabolic boundary Harnack principles via Harnack inequality admitting exceptional set

The Harnack inequality is one of the most fundamental inequalities for positive harmonic functions. Various generalizations of Harnack inequality have been developed for elliptic and parabolic equations. We observe a different type extension for classical harmonic functions, i.e., even if there is a small exceptional set in the Harnack chain, the same Harnack inequality still holds. This extended Harnack inequality has applications to elliptic boundary Harnack principle and parabolic boundary Harnack principle (intrinsically ultracontractivity) for a very nasty domain.

Shin-ichi Ohta (Osaka University)

On weighted Ricci curvature with negative dimension parameter

The weighted Ricci curvature is a modification of the Ricci curvature for a Riemannian manifold equipped with a measure which can be different from the volume measure. Geometric and analytic studies using the weighted Ricci curvature go back to Lichnerowicz' splitting theorem and Bakry-Emery's curvature-dimension condition, in the latter the dimension corresponds to a real parameter appearing in the weighted Ricci curvature. In this talk, we review the recent development of comparison geometry and geometric analysis of weighted Ricci curvature with negative dimension parameter; in this case we have a weaker curvature bound and can cover a wider class of spaces.

Michel Lapidus* (University of California, Riverside)

An introduction to the theory of complex dimensions

We will give some sample results from the new higher-dimensional theory of complex fractal dimensions developed jointly with Goran Radunovic and Darko Zubrinic in the research monograph [2]. We will also explain its connections with the earlier one-dimensional theory of complex dimensions developed, in particular, in the research monograph [1].

In particular, to an arbitrary compact subset A of the N-dimensional Euclidean space (or, more generally, to any relative fractal drum), we will associate new distance and tube zeta functions, as well as discuss their basic properties, including their holomorphic and meromorphic extensions, and the nature and distribution of their poles (or 'complex dimensions'). We will also show that the abscissa of convergence of each of these fractal zeta functions coincides with the upper box (or Minkowski) dimension of the underlying compact set A, and that the associated residues are intimately related to the (possibly suitably averaged) Minkowski content of A. Examples of classical fractals and their complex dimensions will be provided.

Finally, if time permits, we will discuss and extend to any dimension the general definition of fractality proposed by the author (and Machiel van Frankenhuijsen) in their earlier work [1], as the presence of nonreal complex dimensions. We will also provide examples of "hyperfractals", for which the 'critical line' $\{\mathrm{Re}(s) = D\}$, where D is the Minkowski dimension, is not only a natural boundary for the associated fractal zeta functions, but also consist entirely of singularities of those zeta functions.

Fractal tube formulas are obtained which enable us to express the intrinsic oscillations of fractal objects in terms of the underlying complex dimensions and the residues of the associated fractal zeta functions. Intuitively, the real parts of the complex dimensions correspond to the amplitudes of the associated "geometric waves", while their imaginary parts correspond to the frequencies of those waves. This is analogous to Riemann's explicit formula in analytic number theory, expressing the counting function of the primes in terms of the underlying zeros of the celebrated Riemann zeta function. These results are used, in particular, to show the sharpness of an estimate obtained for the abscissa of meromorphic convergence of the spectral zeta functions of fractal drums. Furthermore, we will also briefly discuss recent joint results in which we obtain general fractal tube formulas in this context (that is, for compact subsets of Euclidean space or for relative fractal drums), expressed in terms of the underlying complex dimensions. We may close with a brief discussion of a few of the many open problems stated at the end of the aforementioned book, [1].

In a series of joint papers with Claire David [3-6], this general theory of complex dimensions has recently been significantly extended and applied to the Weierstrass nowhere differentiable function (and a large class of other fractal curves, including the Koch snowflake curve) in order to obtain a corresponding fractal tube formula expressed in terms of the underlying complex dimensions, as well as the associated fractal cohomology. However, in this talk, we will not have the time to expound about the latter work.

[1] Michel L. Lapidus and Machiel van Frankenhuijsen. Fractal Geometry, Complex Dimensions and Zeta Functions: Geometry and Spectra of Fractal Strings. 2013. Springer Monographs in Mathematics, Springer-Verlag, New York; 2nd rev. and enl. edn. of the 2006 edn.

[2] Michel L. Lapidus, Goran Radunovic and Darko Zubrinic. Fractal Zeta Functions and Fractal Drums: Higher Dimensional Theory of Complex Dimensions. 2017. Springer Monographs in Mathematics, Springer International Publishing, Switzerland.

[3] Claire David and Michel L. Lapidus. Weierstrass fractal drums – I – A glimpse of complex dimensions. 2022. hal-03642326, Sorbonne Université, CNRS, Paris, France.

[4] Claire David and Michel L. Lapidus. Weierstrass fractal drums – II – Towards a fractal cohomology. 2022. hal-03758820v3, Sorbonne Université, CNRS, Paris, France.

[5] Claire David and Michel L. Lapidus. Fractal cohomology and complex dimensions of the Weierstrass curve. 2022. hal-03797595v2, Sorbonne Université, CNRS, Paris, France.

[6] Claire David and Michel L. Lapidus. Iterated fractal drums – some new perspectives: Polyhedral measures, atomic decompositions and Morse theory. 2023. hal-039446104, Sorbonne Université, CNRS, Paris, France.

Ben Hambly (University of Oxford)

Spectral asymptotics for some domains with random fractal boundaries

We consider the eigenvalue counting function on fractals and domains with fractal boundary. In the classical case this function is known to have an asymptotic expansion where the leading term is proportional to the volume of the domain and, if the boundary is sufficiently smooth, a second order term proportional to the volume of the boundary. We will focus on the situation of random fractals and random boundaries arising from random recursive constructions. Using limit theorems for the general branching process we will show what happens to this asymptotic expansion in the random case and give examples where the fluctuations due to the randomness are visible in the spectral asymptotics.

Laurent Saloff-Coste (Cornell University)

Limit theorems for stable-like random walks on nilpotent groups

This talk reports on joint work with Z-Q. Chen, T. Kumagai, J. Wang and T. Zheng. I will explain operator limit theorems for random walks on finitely generated nilpotent groups that take stable-like long jumps in various directions. The limit processes are Lévy processes on certain nilpotent Lie groups. An important aspect is that the Lie group which carries the limit process depends both on the original discrete group carrying the random walk and on the stable parameters of the random walk.

Jeremy Tyson (University of Illinois Urbana-Champaign)

On H-type and polarizable Carnot groups

This talk concerns nonlinear potential theory in stratified Lie groups (Carnot groups) and analytic characterizations for a class of well-behaved Carnot groups of step two. According to a classical result of Folland, the fundamental solution for the Laplacian in the nth Heisenberg group is a multiple of N^{2-Q}, where N denotes Korányi's homogeneous norm and $Q = 2n + 2$ is the homogeneous dimension. Capogna, Danielli and Garofalo later observed that an analogous statement is true for the p-Laplacian for every $1 < p < \infty$ and in every Kaplan H-type group, to wit, the fundamental solution of the p-Laplacian is a multiple of $N^{(p-Q)/(p-1)}$ (or $\log(1/N)$ if $p = Q$). Balogh and Tyson introduced the class of polarizable groups as the largest class of Carnot groups equipped with a similar one-parameter family of fundamental solutions for the p-Laplacians, all defined in terms of a fixed homogeneous norm. These groups also support a polar coordinate decomposition using horizontal radial curves; such a decomposition allows for the explicit computation of moduli of ring domains and systems of measures, with applications to the Hölder regularity of quasiconformal mappings. We conjecture that the only polarizable groups are Kaplan H-type groups. Towards this end, we seek to understand the class of H-type groups geometrically within the class of all step two Carnot groups (equipped with a coherent Riemannian metric). We introduce and study a numeric measurement which quantifies the degree to which a given step two Carnot group fails to be of H-type. As an application, we provide several characterizations for H-type groups in terms of analytic identities satisfied by Folland's norm.

Alexander Teplyaev (University of Connecticut)

BV and harmonic functions on fractals

The talk will present the first steps aimed to understand bounded variation functions on self-similar fractals and related estimates of harmonic functions and flows. This is part of a long-term joint project with Patricia Alonso-Ruiz, Fabrice Baudoin, Li Chen, Luke Rogers, and Nageswari Shanmugalingam. We build on the classical pioneering work of Barlow, Bass, Fukushima, Kigami, Kusuoka, and Perkins on Dirichlet and resistance forms, and heat kernel estimates on fractals. In particular, we relate Besov critical exponents and Hölder continuity of harmonic functions with the recently introduced topological-Hausdorff dimension of fractals.

John Mackay (University of Bristol)

Poincaré inequalities and hyperbolic groups

One way to view a Poincaré inequality on (say) a graph is as a tool to measure how well connected the graph is. I will discuss a family of invariants based on these inequalities which generalise the "separation profile" of Benjamini-Schramm-Timar. In the case of Gromov hyperbolic groups, these invariants relate to analytic properties of the boundary at infinity, such as Poincaré inequalities in the sense of Heinonen-Koskela, and Pansu's conformal dimension, and have applications to embedding problems. This is joint work with David Hume and Romain Tessera.

Kôhei Sasaya (Kyoto University)

A system of dyadic cubes of a complete, doubling, uniformly perfect metric space without detours

In this talk, we consider an extension of the notion of the standard dyadic cubes in the Euclidean space to which in a general metric space in Christ's way. We will construct a system of "dyadic cubes" of a complete, doubling, and uniformly perfect metric space, such that any two points in the space are connected by a chain of up to three cubes whose diameters are comparable to the distance of the points.

Ryokichi Tanaka (Kyoto University)

Non-noise sensitivity for random walks on word hyperbolic groups

The noise sensitivity question for random walks on groups asks the following: Given a sample of a random walk with independent increments, produce another by refreshing each increment with an independent copy with a small probability (noise) or retain it otherwise; are these two walks asymptotically independent? We discuss the ℓ^1-noise sensitivity which uses the total variation norm to measure the distance against the distributions of a pair of independent copies, introduced by Benjamini and Brieussel '19. We show that a broad class of random walks on non-elementary word hyperbolic groups are not ℓ^1-noise sensitive in a strong sense, explaining how Poisson boundaries and the dimension of harmonic measures are related to the problem.

Zhen-Qing Chen (University of Washington)

Convergence of effective resistances on generalized Sierpinski carpets

Let F be a generalized Sierpinski carpet inside a d-dimensional unit hypercube with $d \geq 2$ and F_n be its n-stage construction. Denote by d_w and $L \geq 3$ the walk dimension and the length scale of the carpet F. Let X^n be the reflected Brownian motion on F_n running at speed $L^{(d_w-2)n}$. In this talk, we show that X^n converges weakly to a Brownian motion on F. We further show that the effective resistance between two opposite faces of F_n with respect to X^n converges to a positive constant as n tends to infinity. This gives a positive answer to an open problem of Barlow and Bass (1990). Based on a joint work with Shiping Cao.

Ryosuke Shimizu (Kyoto University)

Non-linear potential theory on the Sierpiński carpet

I will provide a review of constructions of $(1, p)$-Sobolev space, self-similar p-energy and p-energy measures on the Sierpiński carpet. For $p = 2$, our self-similar 2-energy and $(1, 2)$-Sobolev space correspond to the canonical self-similar Dirichlet form on the Sierpiński carpet given by Barlow–Bass (1989)/Kusuoka–Zhou (1992). I will also explain the connection between p-energy and the notion of Ahlfors regular conformal dimension.

Eero Saksman (University of Helsinki)

GFF, chaos and Riemann zeta

We survey some recent results (mostly due to others) on probabilistic properties of the Riemann zeta function on the critical line, including connections to Gaussian free field (GFF) and Gaussian multiplicative chaos (GMC).

Grégory Miermont (École Normale Supérieure de Lyon)

Compact Brownian surfaces

We describe the compact scaling limits of uniformly random quadrangulations with boundaries on a surface of arbitrary fixed genus. These limits, called Brownian surfaces, are homeomorphic to the surface of the given genus with or without boundaries depending on the scaling regime of the boundary perimeters of the quadrangulation. They are constructed by appropriate gluings of pieces derived from Brownian geometrical objects (the Brownian plane and half-plane). In this talk, I will review their definition and discuss possible aleternative constructions. This is based on joint work with Jérémie Bettinelli.

Takashi Kumagai (Waseda University)

Spectral dimension of simple random walk on random media

We give a general method that implies on-diagonal heat kernel bounds, in particular, the lower bounds on stationary random media that may have long-range correlations. As an application, we determine the spectral dimension of simple random walk on the long-range percolation model in $\mathbb{Z}^d$. This is joint work with Van Hao Can (Hanoi) and David A. Croydon (Kyoto).